

# Finite codimension stability of some time-periodic hyperbolic equations (via compact resolvents)

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**Abstract:** We identify a class of time-periodic linear symmetric hyperbolic equations that are finite codimension stable, because an associated operator has compact resolvent, sufficiently far to the right in the complex plane. This paper is an attempt to capture abstractly the observation in numerical general relativity that some discretely self-similar spacetimes, such as Choptuik's critical spacetime, are finite codimension stable.

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## 1 Introduction

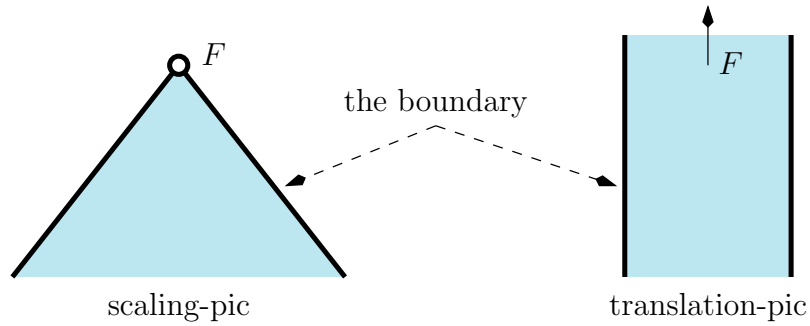
### Motivation

This paper is motivated by a question in general relativity: *How does one explain the numerical observation that some discretely self-similar spacetimes, such as Choptuik's, are stable with small finite codimension [1], [2]?* One might hope for an explanation that is abstract, and neither requires general relativity nor depends on details of spacetimes.

It is plausible that many aspects of stability are already present at the linearized level, and this paper is about linear equations only.

We identify simple abstract assumptions for a time-periodic linear symmetric hyperbolic equation that imply that it is finite codimension stable, roughly: if in the space of solutions one mods out the decaying solutions, only finitely many independent growing solutions are left. This is due to an effect at high frequencies, whereas to actually determine the codimension, one would need a more detailed understanding of low frequencies<sup>1,2</sup>.

This introduction is informal; the logical development starts in Section 3. We are on an infinite solid cylinder,  $\mathbb{R} \times (\text{closed } n\text{-dim ball})$ . It is useful to have two different but diffeomorphic pictures of this cylinder:



In both, time increases upwards; space is horizontal; horizontal cross-sections are closed  $n$ -dim balls. Here  $F \notin \text{cylinder}$  is the future limit point. Let  $\mathbf{T}$  be a fixed self-diffeomorphism of the cylinder, equivalently given:

- In the scaling-pic, by a scaling about and towards  $F$ .
- In the translation-pic, by a translation upwards.

Consider a linear symmetric hyperbolic operator  $\widehat{\mathbf{D}}$  on the cylinder. That is, an  $N \times N$  matrix whose entries are first order differential operators, subject to a standard condition on its principal part. We assume:

- *$\mathbf{T}$ -periodicity:*  $\widehat{\mathbf{D}}$  commutes with composition by  $\mathbf{T}$ , that is, for all functions  $\widehat{u} : \text{cylinder} \rightarrow \mathbb{C}^N$  one has  $(\widehat{\mathbf{D}}\widehat{u}) \circ \mathbf{T} = \widehat{\mathbf{D}}(\widehat{u} \circ \mathbf{T})$ .
- *Causal independence:*  $\widehat{\mathbf{D}}$  allows information to flow out through the boundary, but no information to flow in, as time increases.

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<sup>1</sup>Our abstract assumptions are consistent with all finite values of the codimension.

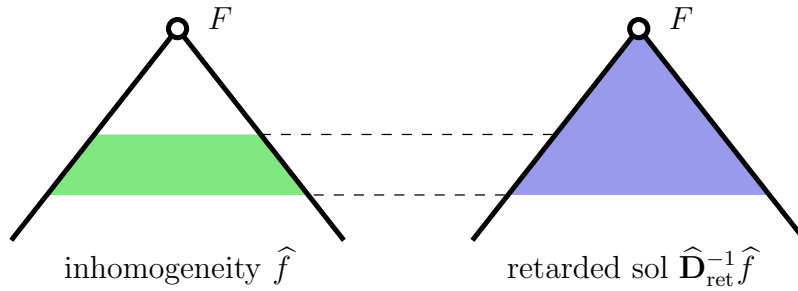
<sup>2</sup>Better understanding low frequencies will often be an effectively finite-dimensional problem, amenable to (say) rigorous computer-assisted study. Whether there is a more conceptual, abstract approach to low frequencies is an interesting question.

This operator  $\widehat{\mathbf{D}}$  could arise in general relativity from linearizing the Einstein equations about a spacetime for which  $\mathbf{T}$  is a constant rescaling of the metric<sup>3</sup>, aka a discretely self-similar spacetime, or it could arise somewhere else.

For every nice function  $\widehat{f} : \text{cylinder} \rightarrow \mathbb{C}^N$  with compact support in time, let  $\widehat{\mathbf{D}}_{\text{ret}}^{-1}\widehat{f} : \text{cylinder} \rightarrow \mathbb{C}^N$  be the unique solution to

$$\widehat{\mathbf{D}}(\widehat{\mathbf{D}}_{\text{ret}}^{-1}\widehat{f}) = \widehat{f}$$

that vanishes in the past, aka the retarded solution. Boundary conditions must not be given, by causal independence forward in time.



A very simple scenario is *finite codimension stability*: That is, when there exist finitely many functions  $\widehat{u}_1, \dots, \widehat{u}_J : \text{cylinder} \rightarrow \mathbb{C}^N$  in the kernel of  $\widehat{\mathbf{D}}$ , and complex valued linear functionals  $\ell_1, \dots, \ell_J$  such that, for all nice  $\widehat{f}$ ,

$$\widehat{\mathbf{D}}_{\text{ret}}^{-1}\widehat{f} - \widehat{u}_1 \ell_1(\widehat{f}) - \dots - \widehat{u}_J \ell_J(\widehat{f})$$

decays towards the future limit point  $F$ . In this paper we identify simple abstract assumptions on  $\widehat{\mathbf{D}}$ , on top of the ones already stated, that imply finite codimension stability.

With a hyperbolic equation, any non-smoothness in  $\widehat{f}$  can persist in the retarded solution, which can easily spoil finite codimension stability. To avoid this, we put ourselves in function spaces of  $\infty$ -differentiable functions<sup>4</sup>.

$\mathbf{T}$ -periodicity is the only symmetry assumption<sup>5</sup>.

<sup>3</sup>Beware that the linearized Einstein equations are not symmetric hyperbolic out of the box. Rather, the space of solutions to say the linearized vacuum Einstein equations, about a background solution, is the first cohomology of a differential graded Lie algebra [4]. The equations can be made symmetric hyperbolic through gauge-fixing.

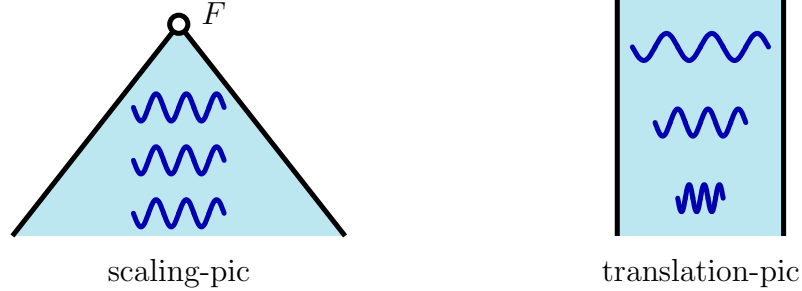
<sup>4</sup>These spaces are consistent with nontrivial functions having compact support. As for dropping  $\infty$ -differentiability, consider Counterexample 2.3 and see Remark 7.4.

<sup>5</sup>In particular, spherical symmetry is not assumed. Note that Choptuik's critical space-time, a discretely self-similar 1 + 3 dim solution to Einstein's equations coupled to a massless scalar field, is spherically symmetric. An analogous solution to the vacuum Einstein equations, if such exists, cannot be spherically symmetric.

This paper leaves open whether its results can be applied to the linearized Einstein equations about Choptuik's critical spacetime. One could start from the proof of existence of this spacetime in [3]. Incidentally, even though [3] is not about stability, the proof of existence there has some technical overlap with this paper, see Example 2.1.

## Intuition

Let  $\hat{u}$  be a high frequency solution to the homogeneous equation  $\hat{\mathbf{D}}\hat{u} = 0$ , and pretend that the solution's wavelength does not change much under propagation, as measured in the scaling-pic. The same solution looks rather different in the translation-pic, where its wavelength increases:



There is a more invariant, picture-independent way of saying this: Pick a horizontal cross-section  $B$ , a ball, take snapshots of the solution on  $\mathbf{T}^p(B)$  for  $p \in \mathbb{N}_0$ , and pull them back to  $B$ . These pullbacks  $\hat{u}|_{\mathbf{T}^p(B)} \circ \mathbf{T}^p$  are a sequence of functions  $B \rightarrow \mathbb{C}^N$  with increasing wavelengths.

Clearly not all symmetric hyperbolic operators behave like this. We impose a simple condition on the principal part of  $\hat{\mathbf{D}}$  that gives such a regularizing effect at high frequencies; this is the assumption later called (iii).

## Outline

We use coordinates in which the cylinder is given by

$$\begin{aligned}\hat{\Omega} &= \{(x^0, \dots, x^n) \in \mathbb{R}^{1+n} \mid (x^1)^2 + \dots + (x^n)^2 \leq 1\} \\ &= \mathbb{R} \times (\text{closed unit ball in } \mathbb{R}^n)\end{aligned}$$

and in which the self-diffeomorphism is given by

$$\mathbf{T} : (x^0, x^1, \dots, x^n) \mapsto (x^0 + 2\pi, x^1, \dots, x^n)$$

Many calculations in this paper are carried out on the compact quotient

$$\begin{aligned}\Omega &= \widehat{\Omega}/\mathbf{T} \\ &= (\mathbb{R}/2\pi\mathbb{Z}) \times (\text{closed unit ball in } \mathbb{R}^n)\end{aligned}$$

There are four abstract assumptions on the operator  $\widehat{\mathbf{D}}$ . For the purpose of this introduction, we state them informally:

- (i)  $\mathbf{T}$ -periodicity and linear symmetric hyperbolicity, with  $x^0$  as time.
- (ii) Causal independence, forward in time.
- (iii) Condition on the principal part, significant at high frequencies.
- (iv) Regularity condition on the operator itself.

The main goal is to construct the right-inverse  $\widehat{\mathbf{D}}_{\text{ret}}^{-1}$ , aka the retarded Green's function, and to show finite codimension stability. By our earlier definition, we must show the existence of a finite-rank<sup>6</sup> operator  $\mathbf{F}$  such that  $\widehat{\mathbf{D}} \circ \mathbf{F} = 0$  and such that all elements in the image of

$$\widehat{\mathbf{D}}_{\text{ret}}^{-1} - \mathbf{F}$$

decay as  $x^0 \rightarrow +\infty$ . Finite codimension stability is Theorem 6.1.

Theorem 6.1 is proved by a contour integration argument. This is based on a Fourier-like transform, used to write the retarded Green's function as a contour integral in terms of the resolvent of

$$\mathbf{D} = \widehat{\mathbf{D}}|_{\mathbf{T}\text{-periodic functions}}$$

where the vertical line means 'restricted to'. The two operators  $\widehat{\mathbf{D}}$  and  $\mathbf{D}$  are obviously equivalent, but it is always understood that:

$$\begin{aligned}\widehat{\mathbf{D}} &\text{ maps functions } \widehat{\Omega} \rightarrow \mathbb{C}^N \text{ to functions } \widehat{\Omega} \rightarrow \mathbb{C}^N \\ \mathbf{D} &\text{ maps functions } \Omega \rightarrow \mathbb{C}^N \text{ to functions } \Omega \rightarrow \mathbb{C}^N\end{aligned}$$

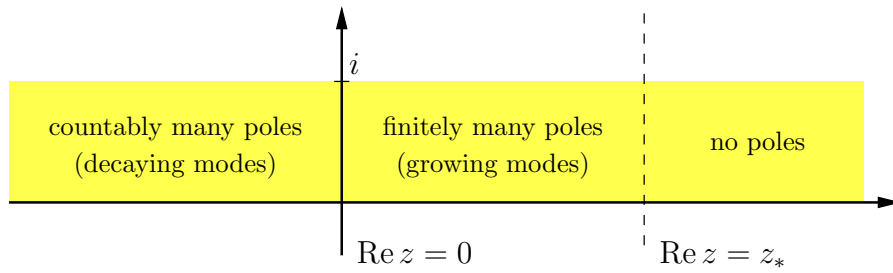
Studying the resolvent of  $\mathbf{D}$  is the main technical task. This resolvent  $(\mathbf{D} + z\mathbb{1})^{-1}$  requires function spaces to make sense, and may not exist for all

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<sup>6</sup>Suppose  $V, V'$  are vector spaces. A linear operator  $V \rightarrow V'$  is finite-rank iff it is the composition of a linear map  $V \rightarrow \mathbb{C}^J$  with a linear map  $\mathbb{C}^J \rightarrow V'$ , for some integer  $J$ .

$z \in \mathbb{C}$ . We study the resolvent on suitable Banach spaces of  $\infty$ -differentiable functions  $\Omega \rightarrow \mathbb{C}^N$  and show that for  $\operatorname{Re} z$  bigger than some constant  $z_* \in \mathbb{R}$ , the resolvent exists and is compact<sup>7</sup>. By compactness at even just one point, the resolvent extends meromorphically to  $\mathbb{C}$ , and the spectral projection associated to each pole is finite-rank, that is, multiplicities are finite.

The resolvent is periodic under  $z \sim z+i$  up to conjugation<sup>8</sup>. In particular,  $z$  is a pole if and only if  $z+i$  is a pole. A fundamental domain such as  $\{z \in \mathbb{C} \mid 0 \leq \operatorname{Im} z < 1\}$  decomposes into three pieces:



For every  $c \in \mathbb{R}$ , only finitely many poles have  $\operatorname{Re} z \geq c$  and  $0 \leq \operatorname{Im} z < 1$ . Therefore only finitely many have  $\operatorname{Re} z \geq 0$  and  $0 \leq \operatorname{Im} z < 1$ ; their spectral projections yield an explicit formula for the finite-rank operator  $\mathbf{F}$ .

One can think of individual poles as decaying or growing modes, but since the spectral theorem for self-adjoint operators on Hilbert spaces does not apply here, we keep away from eigendecompositions.

## 2 Examples and counterexamples

Before stating our abstract assumptions on  $\mathbf{D}$ , we discuss examples. They are invariant under arbitrary translations in  $x^0$ , hence trivially  $\mathbf{T}$ -periodic.

Let  $\partial_0, \dots, \partial_n$  be the partial derivatives,  $\partial_i x^j = \delta_i^j$ .

*Example 2.1.* In  $n = 1$  and  $N = 1$  consider

$$\mathbf{D} = \partial_0 + \mu(x^1 - x_*^1)\partial_1$$

with  $\mu > 0$  and  $-1 \leq x_*^1 \leq 1$ . It satisfies all abstract assumptions. Such operators are used in [3], where the inverse of  $\partial_0 + \mu(x^1 + 1)\partial_1 + \mu$  is calculated

<sup>7</sup>Suppose  $V, V'$  are Banach spaces. A linear map  $V \rightarrow V'$  is compact iff every bounded sequence in  $V$  is mapped to a sequence in  $V'$  that contains a Cauchy subsequence.

<sup>8</sup>Just like the wave vector of a ‘Bloch wave’ on a lattice is defined only modulo the reciprocal lattice. Here the lattice is  $x^0 \sim x^0 + 2\pi$ , the reciprocal lattice is  $z \sim z + i$ .

explicitly, on some space of real analytic functions on  $\Omega = (\mathbb{R}/2\pi\mathbb{Z}) \times [-1, 1]$ , using Fourier-Chebyshev series. This inverse has regularizing features at high frequencies, used in [3] to reduce the proof of existence of Choptuik's space-time to a finite, if very large, computer calculation for the low frequencies. The regularizing features of this inverse are similar to the estimates on the resolvent of  $\mathbf{D}$  obtained in this paper, in a more general setting.

*Example 2.2.* In  $n = 3$  and  $N = 2$  consider

$$\mathbf{D} = \begin{pmatrix} \partial_0 + \mu\partial_3 & \mu\partial_1 + i\mu\partial_2 \\ \mu\partial_1 - i\mu\partial_2 & \partial_0 - \mu\partial_3 \end{pmatrix} + \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} (x^1\partial_1 + x^2\partial_2 + x^3\partial_3)$$

with  $\mu > 0$ . It satisfies all abstract assumptions.

In Examples 2.1 and 2.2, consider the eigenvalue problem  $\ker(\mathbf{D} + z\mathbb{I}) \neq 0$  with  $z \in \mathbb{C}$ . The point is that every eigenfunction  $u : \Omega \rightarrow \mathbb{C}^N$ , or rather its lift  $\hat{u} : \hat{\Omega} \rightarrow \mathbb{C}^N$ , yields a homogeneous solution,  $\hat{\mathbf{D}}(e^{zx^0}\hat{u}) = 0$ . There are real analytic eigenfunctions of the form

$$e^{iqx^0} P(x^1, \dots, x^n)$$

where  $q \in \mathbb{Z}$ , where  $P$  is an  $N$ -component polynomial of total degree  $p \in \mathbb{N}_0$ , and the eigenvalue is  $z = -iq - \mu p$ . In  $P$  one can freely choose all terms of homogeneous degree  $p$ , and then all terms of degree less than  $p$  are recursively determined. The set  $i\mathbb{Z} - \mu\mathbb{N}_0$  of such eigenvalues is discrete, a half-lattice, periodic under  $z \sim z + i$ , and has real parts bounded from above.

*Counterexample 2.3* (eigenfunctions not  $\infty$ -differentiable). The operator in Example 2.1 has other eigenfunctions such as  $\max\{0, x^1 - x_*^1\}^s$  for all  $s \in \mathbb{C}$  with say  $\operatorname{Re} s > 1$  and eigenvalue  $z = -\mu s$ . The set of such eigenvalues is not discrete<sup>9</sup>. In this paper we avoid such eigenvalues by putting ourselves in suitable Banach spaces of  $\infty$ -differentiable functions.

*Counterexample 2.4* (no causal independence). Take Example 2.1 with  $x_*^1 < -1$ . It satisfies all abstract assumptions except **(ii)**. Since  $x^1 - x_*^1 > 0$  everywhere on  $\Omega = (\mathbb{R}/2\pi\mathbb{Z}) \times [-1, 1]$ , there are real analytic eigenfunctions  $(x^1 - x_*^1)^s$  for all  $s \in \mathbb{C}$ , with  $z = -\mu s$ .

*Counterexample 2.5* (no high frequency effect). Consider  $\mathbf{D} = \partial_0$ . It satisfies all abstract assumptions except **(iii)**. It features infinite multiplicities even for real analytic eigenfunctions.

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<sup>9</sup>Such eigenvalues can end up in the right half-plane, say for operators of the form  $\mathbf{D} = \partial_0 + \mu x^1 \partial_1 + \text{const}$ , with  $\mu > 0$ , which also satisfy the abstract assumptions.

### 3 Abstract assumptions

Let  $n \geq 1$  be an integer. Our working domain and its boundary are:

$$\begin{aligned}\Omega &= \{(x^0, x^1, \dots, x^n) \in (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^n \mid (x^1)^2 + \dots + (x^n)^2 \leq 1\} \\ \partial\Omega &= \{(x^0, x^1, \dots, x^n) \in (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^n \mid (x^1)^2 + \dots + (x^n)^2 = 1\}\end{aligned}$$

Here  $x^0 \sim x^0 + 2\pi$ . The partial derivatives  $\partial_0, \partial_1, \dots, \partial_n$  satisfy  $\partial_i x^j = \delta_i^j$ . Along  $\partial\Omega$ , the outward unit normal is  $(\omega_0, \omega_1, \dots, \omega_n) = (0, x^1, \dots, x^n)$ .

Given and fixed is a differential operator<sup>10,11</sup>

$$\mathbf{D} = A^i \partial_i + B$$

subject to four abstract assumptions:

- (i)  $A^0, \dots, A^n, B$  are  $N \times N$  matrices whose entries are  $\infty$ -differentiable functions  $\Omega \rightarrow \mathbb{C}$ . The matrices  $A^0, \dots, A^n$  are Hermitian and  $A^0$  is positive definite ( $\dagger$  is the conjugate transpose):

$$\begin{aligned}(A^i)^\dagger &= A^i \\ A^0 &> 0\end{aligned}$$

- (ii)  $A^i \omega_i$  is positive semidefinite,  $A^i \omega_i \geq 0$ , along the boundary  $\partial\Omega$ .

- (iii) Set

$$A^{ij} = \frac{1}{2}(\partial_i A^j + \partial_j A^i)$$

There is a constant  $\xi > 0$  and there are  $N \times N$  matrices  $\Xi^0, \dots, \Xi^n$  whose entries are  $\infty$ -differentiable functions  $\Omega \rightarrow \mathbb{C}$  such that

$$w_i^\dagger (\xi A^{ij} + \frac{1}{2} A^i \Xi^j + \frac{1}{2} (\Xi^i)^\dagger A^j) w_j \geq \delta^{ij} w_i^\dagger w_j$$

for all  $w_0, \dots, w_n \in \mathbb{C}^N$  and everywhere on  $\Omega$ .

- (iv) There is a sequence of constants  $q_0, q_1, q_2, \dots \geq 0$  with  $q_0 = 1$  and

$$q_{k+\ell} \leq q_k q_\ell$$

for all  $k, \ell \in \mathbb{N}_0$ , and there is a constant  $Q > 0$  such that

$$\begin{aligned}\sum_{k=K+1}^{\infty} (k+1)^{n/2} \frac{q_{k-1} |A|_k}{k!} &\leq Q q_K \\ \sum_{k=K}^{\infty} (k+1)^{n/2} \frac{q_k |B + z A^0|_k}{k!} &\leq Q q_K (1 + |z|)\end{aligned}$$

for  $K = 0, 1$  (two values only) and for all  $z \in \mathbb{C}$ .

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<sup>10</sup>As usual,  $A^i \partial_i = A^0 \partial_0 + \dots + A^n \partial_n$ .

<sup>11</sup>For simplicity, in Sections 1 and 2 we used  $\mathbb{1}$  in places where we use  $A^0$  from now on.



In **(iv)** we use the following supremum norms of derivatives of order  $k \in \mathbb{N}_0$ :

$$|A|_k = \sup_{\Omega} \sup_{\substack{w_{\alpha i} \in \mathbb{C}^N \\ \text{not all zero}}} \frac{\left| \sum_{|\alpha|=k} \sum_{i=0}^n \frac{k!}{\alpha!} (\partial^\alpha A^i) w_{\alpha i} \right|_{\mathbb{C}^N}}{\left( \sum_{|\alpha|=k} \sum_{i=0}^n \frac{k!}{\alpha!} |w_{\alpha i}|_{\mathbb{C}^N}^2 \right)^{1/2}}$$

$$|B + zA^0|_k = \sup_{\Omega} \sup_{\substack{w_{\alpha} \in \mathbb{C}^N \\ \text{not all zero}}} \frac{\left| \sum_{|\alpha|=k} \frac{k!}{\alpha!} (\partial^\alpha (B + zA^0)) w_{\alpha} \right|_{\mathbb{C}^N}}{\left( \sum_{|\alpha|=k} \frac{k!}{\alpha!} |w_{\alpha}|_{\mathbb{C}^N}^2 \right)^{1/2}}$$

Throughout this paper,  $\alpha \in \mathbb{N}_0^{1+n}$  is a multi-index, with  $|\alpha| = \alpha_0 + \dots + \alpha_n$ ,  $\alpha! = \alpha_0! \dots \alpha_n!$ ,  $\partial^\alpha = (\partial_0)^{\alpha_0} \dots (\partial_n)^{\alpha_n}$ . Also,  $|w|_{\mathbb{C}^N} = (w^\dagger w)^{1/2}$  if  $w \in \mathbb{C}^N$ .

Along with **D** itself, we consider  $\xi, \Xi^i, Q, q_\ell$  to be part of the data. That is, they are parameters of the theorems in this paper.

- Some theorems require  $q_\ell > 0$  for all  $\ell$ .
- Some theorems impose a smallness condition on  $q_1$ . It is easy to satisfy both this smallness condition and **(iv)** simultaneously, because of the following fact: If **(iv)** holds for a sequence  $(q_\ell)$  then it continues to hold for the sequence  $(\kappa^\ell q_\ell)$  for all  $0 < \kappa \leq 1$ , with the same  $Q$ .

*Informal discussion.* Assumption **(i)** makes **D** symmetric hyperbolic, in the sense of K.O. Friedrichs. Here  $A^0 > 0$  holds uniformly, since  $\Omega$  is compact. Assumption **(ii)** is causal independence forward in time. Assumption **(iii)** is a positivity condition for the ‘deformation tensor’  $A^{ij}$ ; the naive condition with  $\Xi^0 = \dots = \Xi^n = 0$  is too strong, in fact inconsistent with **(i)**<sup>12</sup>.

Assumption **(iv)** are bounds for  $A^i$  and  $B$ . Two extreme cases are:

- $q_0 = 1$  and  $q_1 \geq 0$  but  $q_2 = q_3 = \dots = 0$ .
- $q_\ell = (q_1)^\ell$  for some  $q_1 > 0$ .

In the last case,  $A^i$  and  $B$  are real analytic. The last case is extreme because **(iv)** requires  $q_\ell \leq (q_1)^\ell$ . Intermediate cases include sequences with  $q_\ell > 0$  for all  $\ell$  that go to zero super-exponentially as  $\ell \rightarrow \infty$ .

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<sup>12</sup>Fix a nonzero  $w \in \mathbb{C}^N$  and then set  $W = \int_0^{2\pi} dx^0 w^\dagger A^{00} w$ , the integral taken along say  $x^1 = \dots = x^n = 0$ . Assumption **(iii)** with  $\Xi^0 = \dots = \Xi^n = 0$  would imply  $W > 0$ , whereas  $A^{00} = \partial_0 A^0$  and periodicity  $x^0 \sim x^0 + 2\pi$  in **(i)** imply  $W = 0$ .

## 4 Main technical theorems, compact resolvent

For all  $u, v : \Omega \rightarrow \mathbb{C}^N$  let  $\langle u, v \rangle = \int_{\Omega} u^\dagger v$  be the standard inner product, and let  $\|u\| = \langle u, u \rangle^{1/2}$ . Define the Sobolev seminorm of order  $\ell \in \mathbb{N}_0$  by

$$\|u\|_\ell = \left( \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} \|\partial^\alpha u\|^2 \right)^{1/2}$$

We use the following function spaces<sup>13,14</sup>:

- $\mathcal{C}^\infty = \{u : \Omega \rightarrow \mathbb{C}^N \mid u \text{ is } \infty\text{-differentiable}\}$ .
- $\mathcal{H}_h = \{u : \Omega \rightarrow \mathbb{C}^N \mid \|u\|_h < \infty\}$  for all  $h \in \mathbb{N}_0$ , with norm

$$\|u\|_h = \sum_{\ell=0}^{\infty} \frac{q_\ell \|u\|_\ell}{(\ell + h)!}$$

The Banach space  $\mathcal{H}_h$  depends implicitly on the sequence  $(q_\ell)$  in **(iv)**.

We assume **(i)**, **(ii)**, **(iii)**, **(iv)**. The theorems below refer to two constants<sup>15</sup>:

- A constant  $z_* \in \mathbb{R}$  that depends only on  $\mathbf{D}$ .
- A constant  $q_{1*} > 0$  that depends only on  $\mathbf{D}, \xi, \Xi^i, Q$ .

For each  $z \in \mathbb{C}$  define<sup>16</sup>

$$\mathbf{D}_z = \mathbf{D} + zA^0$$

**Theorem 4.1.** *If  $\operatorname{Re} z \geq z_*$ , then  $\mathbf{D}_z : \mathcal{C}^\infty \rightarrow \mathcal{C}^\infty$  is bijective.*

**Theorem 4.2.** *If  $\operatorname{Re} z \geq z_*$  and  $q_1 \leq q_{1*}$ , then*

$$\mathbf{D}_z^{-1}(\mathcal{C}^\infty \cap \mathcal{H}_1) \subseteq \mathcal{C}^\infty \cap \mathcal{H}_0$$

*and it extends uniquely to a bounded linear map  $\mathbf{D}_z^{-1} : \mathcal{H}_1 \rightarrow \mathcal{H}_0$ .*

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<sup>13</sup>By definition, a function  $\Omega \rightarrow \mathbb{C}^N$  is  $\infty$ -differentiable if and only if it is the restriction of an  $\infty$ -differentiable function  $(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^n \rightarrow \mathbb{C}^N$ .

<sup>14</sup>If  $q_\ell > 0$  for all  $\ell$ , then  $\mathcal{H}_h \subseteq \mathcal{C}^\infty$ . If  $q_\ell = 0$  for one  $\ell$  and therefore for almost all  $\ell$ , then  $\mathcal{H}_h$  is a Sobolev space and  $\mathcal{C}^\infty \subseteq \mathcal{H}_h$ . In either case,  $\mathcal{C}^\infty \cap \mathcal{H}_h \subseteq \mathcal{H}_h$  is dense.

<sup>15</sup>Their explicit values are in Lemmas 5.3 and 5.16.

<sup>16</sup>The operator  $\mathbf{D}_z$  is used to construct the retarded Green's function, in Section 6.

**Theorem 4.3.** *If  $\operatorname{Re} z \geq z_*$  and  $q_1 \leq q_{1*}$  and  $q_\ell > 0$  for all  $\ell$ , then*

$$\mathbf{D}_z^{-1} : \mathcal{H}_1 \rightarrow \mathcal{H}_1 \text{ is compact}$$

*and there exists a unique extension to a meromorphic map  $w \mapsto \mathbf{D}_w^{-1}$  from  $\mathbb{C}$  to the Banach space of bounded linear maps  $\mathcal{H}_1 \rightarrow \mathcal{H}_1$ . This extension satisfies the first resolvent identity and is periodic up to conjugation<sup>17</sup>:*

$$\begin{aligned} \mathbf{D}_w^{-1} - \mathbf{D}_{w'}^{-1} &= -(w - w')\mathbf{D}_w^{-1}A^0\mathbf{D}_{w'}^{-1} \\ \mathbf{D}_{w+i}^{-1} &= e^{-ix^0}\mathbf{D}_w^{-1}e^{ix^0} \end{aligned}$$

*For all  $w \in \mathbb{C}$  away from poles,  $\mathbf{D}_w^{-1}$  is compact.*

Theorem 4.1 follows from Corollary 5.5 and Lemma 5.20. Theorem 4.2 then follows from Lemma 5.16.

If  $q_\ell > 0$  for all  $\ell$ , then the inclusion  $\mathcal{H}_0 \hookrightarrow \mathcal{H}_1$  is compact by Lemma 5.2, and hence Theorem 4.2 implies compactness of  $\mathbf{D}_z^{-1}$  in Theorem 4.3. In turn, compactness implies the unique meromorphic extension: it can be explicitly defined by fixing any  $z \in \mathbb{C}$  with  $\operatorname{Re} z \geq z_*$  and setting

$$\mathbf{D}_w^{-1} = \mathbf{D}_z^{-1}(\mathbb{1} + (w - z)A^0\mathbf{D}_z^{-1})^{-1}$$

which is meromorphic in  $w \in \mathbb{C}$  by the spectral theory of compact operators, applied to  $A^0\mathbf{D}_z^{-1} : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ . This definition is consistent when  $\operatorname{Re} w \geq z_*$ .

## 5 Lemmas and proofs

**Lemma 5.1.** *Suppose  $q_\ell > 0$  for all  $\ell$ . Then for all  $\varepsilon > 0$ , every sequence in  $\mathcal{H}_0$  with  $\mathcal{H}_0$ -diameter  $\leq 1$  has a subsequence with  $\mathcal{H}_1$ -diameter  $\leq \varepsilon$ . Here the diameter is the supremum of all pairwise distances.*

*Proof.* Pick an integer  $k \geq 1$  big enough to make  $\frac{1}{k+1} \leq \frac{\varepsilon}{2}$ . By  $q_\ell > 0$  for all  $\ell$ , the given sequence in  $\mathcal{H}_0$  is bounded in the Sobolev space of order  $k$ . Hence it has a Cauchy subsequence in the Sobolev space of order  $k - 1$ , by Rellich's theorem<sup>18</sup>. In particular, it has a subsequence  $(u_p)_{p \geq 0}$  such that

$$\sum_{\ell=0}^{k-1} \frac{q_\ell \|u_p - u_q\|_\ell}{(\ell + 1)!} \leq \frac{\varepsilon}{2}$$

---

<sup>17</sup>The three operators  $A^0, e^{\pm ix^0} : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  are bounded by Lemmas 5.17 and 5.18.

<sup>18</sup>The theorem applies because  $\Omega$  is compact.

for all  $p, q$ . On the other hand,

$$\sum_{\ell=k}^{\infty} \frac{q_{\ell} \|u_p - u_q\|_{\ell}}{(\ell+1)!} \leq \frac{1}{k+1} \|u_p - u_q\|_0 \leq \frac{1}{k+1} \leq \frac{\varepsilon}{2}$$

for all  $p, q$ . Therefore  $\|u_p - u_q\|_1 \leq \varepsilon$  for all  $p, q$  as required.

**Lemma 5.2.** *Suppose  $q_{\ell} > 0$  for all  $\ell$ . Then  $\mathcal{H}_0 \hookrightarrow \mathcal{H}_1$  is compact.*

*Proof.* Given is a sequence in  $\mathcal{H}_0$  with  $\mathcal{H}_0$ -diameter  $\leq 1$ . A subsequence with  $\mathcal{H}_1$ -diameter  $\leq \varepsilon$  will be called an  $\varepsilon$ -subseq; an  $\varepsilon$ -subseq exists for all  $\varepsilon > 0$  by Lemma 5.1. First pick a  $\frac{1}{2}$ -subseq. Given this  $\frac{1}{2}$ -subseq, store its first element and pick a  $\frac{1}{3}$ -subseq of the rest. Given this  $\frac{1}{3}$ -subseq, store its first element and pick a  $\frac{1}{4}$ -subseq of the rest. And so forth. The subsequence of stored elements is a Cauchy sequence relative to  $\mathcal{H}_1$ .

**Lemma 5.3.** *There exist constants  $z_* \in \mathbb{R}$  and  $R > 0$  such that if  $\operatorname{Re} z \geq z_*$  then for all  $u \in \mathcal{C}^{\infty}$  the functions  $J^i = \frac{1}{2} u^{\dagger} A^i u$  satisfy both*

$$\begin{aligned} \partial_i J^i &\leq -J^0 + \operatorname{Re}(u^{\dagger} \mathbf{D}_z u) \\ \partial_i J^i &\leq -R_z(u^{\dagger} u) + \operatorname{Re}(u^{\dagger} \mathbf{D}_z u) \end{aligned}$$

*everywhere on  $\Omega$ , where  $R_z = R(1 + |\operatorname{Re} z|)$ . Convention: All occurrences of  $z_*$  and  $R$  in this paper refer to their values as determined by this lemma.*

*Proof.* Using (i) we have  $\partial_i J^i = -u^{\dagger} K_z u + \operatorname{Re}(u^{\dagger} \mathbf{D}_z u)$  where

$$K_z = \frac{1}{2}(-\partial_i A^i + B + B^{\dagger}) + (\operatorname{Re} z) A^0$$

is Hermitian. Since  $A^0 > 0$  uniformly, we can choose  $z_*$  such that  $\operatorname{Re} z \geq z_*$  implies  $K_z \geq \frac{1}{2} A^0$ , and choose  $R > 0$  such that  $\operatorname{Re} z \geq z_*$  implies  $K_z \geq R_z \mathbb{1}$ .

**Lemma 5.4.** *If  $\operatorname{Re} z \geq z_*$  then*

$$\langle u, u \rangle \leq \frac{1}{R_z} \operatorname{Re} \langle u, \mathbf{D}_z u \rangle$$

*for all  $u \in \mathcal{C}^{\infty}$ .*

*Proof.* By the divergence theorem and by (ii),

$$\int_{\Omega} \partial_i J^i = \int_{\partial\Omega} J^i \omega_i \geq 0$$

Now use the second inequality in Lemma 5.3.

**Corollary 5.5.** *If  $\operatorname{Re} z \geq z_*$  then  $\mathbf{D}_z : \mathcal{C}^\infty \rightarrow \mathcal{C}^\infty$  is injective.*

The next lemma is where  $\partial_j A^i$  comes out of hiding; recall that its symmetric part is the deformation tensor in **(iii)**. The lemma splits the commutator  $[\partial^\alpha, \mathbf{D}_z]$  into two parts: one part with derivatives of order  $|\alpha|$ , the other part with derivatives of order less than  $|\alpha|$ , aka lower order terms (lot).

**Lemma 5.6.** *Let  $e_i \in \mathbb{N}_0^{1+n}$  be the  $i$ -th unit vector. We have<sup>19</sup>*

$$[\partial^\alpha, \mathbf{D}_z] = \alpha_j (\partial_j A^i) \partial^{\alpha+e_i-e_j} + [\partial^\alpha, \mathbf{D}_z]_{\text{lot}}$$

where, by definition,

$$[\partial^\alpha, \mathbf{D}_z]_{\text{lot}} = \sum_{\substack{\beta \leq \alpha \\ |\beta| \geq 2}} \binom{\alpha}{\beta} (\partial^\beta A^i) \partial^{\alpha-\beta+e_i} + \sum_{\substack{\beta \leq \alpha \\ |\beta| \geq 1}} \binom{\alpha}{\beta} (\partial^\beta (B + zA^0)) \partial^{\alpha-\beta}$$

*Proof.* The product rule.

**Lemma 5.7.** *For all  $u \in \mathcal{C}^\infty$  and  $\ell \in \mathbb{N}_0$  we have*

$$\begin{aligned} \operatorname{Re} \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} u_\alpha^\dagger ([\partial^\alpha, \mathbf{D}_z] - [\partial^\alpha, \mathbf{D}_z]_{\text{lot}}) u \\ \geq \frac{\ell}{\xi} \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} u_\alpha^\dagger u_\alpha - \frac{\ell}{\xi} \operatorname{Re} \sum_{|\beta|=\ell-1} \frac{(\ell-1)!}{\beta!} (\Xi^i u_{\beta+e_i})^\dagger (A^j u_{\beta+e_j}) \end{aligned}$$

where  $u_\alpha = \partial^\alpha u$ . This lemma uses notation from **(iii)**.

*Proof.* On the left hand side, use Lemma 5.6, substitute  $\alpha = \beta + e_j$ , use  $\alpha_j \frac{1}{\alpha!} = \frac{1}{\beta!}$  and  $\operatorname{Re}(u_{\beta+e_j}^\dagger (\partial_j A^i) u_{\beta+e_i}) = u_{\beta+e_i}^\dagger A^{ij} u_{\beta+e_j}$ ; the last because  $A^i$  is Hermitian by **(i)**. One finds that the left hand side is equal to

$$\ell \sum_{|\beta|=\ell-1} \frac{(\ell-1)!}{\beta!} u_{\beta+e_i}^\dagger A^{ij} u_{\beta+e_j}$$

Now the lemma follows from **(iii)** and the combinatorial Lemma 5.8.

**Lemma 5.8.** *Let  $c_\alpha \in \mathbb{C}$  be a collection of complex numbers, where  $\alpha$  runs over  $|\alpha| = \ell$  for some integer  $\ell \geq 1$ . Then*

$$\sum_{|\beta|=\ell-1} \frac{(\ell-1)!}{\beta!} \sum_{i=0}^n c_{\beta+e_i} = \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} c_\alpha$$

---

<sup>19</sup>It is clear from context where summation over  $i = 0 \dots n$  and/or  $j = 0 \dots n$  is implicit.

*Proof.* It suffices to check this for  $c_\alpha = \delta_{\alpha=\gamma}$  for all fixed  $|\gamma| = \ell$ . That is, all we need is  $\sum_{i=0}^n \frac{(\ell-1)!}{(\gamma-e_i)!} = \frac{\ell!}{\gamma!}$ , which is true, and the lemma is proved.

**Lemma 5.9.** *Let  $a_\alpha, b_\alpha, c_{\alpha k} \in \mathcal{C}^\infty$  be collections of functions, where  $\alpha$  runs over  $|\alpha| = \ell \in \mathbb{N}_0$ , and where  $k$  runs over a finite index set. Then*

$$\begin{aligned} \left| \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} \langle a_\alpha, b_\alpha \rangle \right| &\leq \left( \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} \|a_\alpha\|^2 \right)^{1/2} \left( \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} \|b_\alpha\|^2 \right)^{1/2} \\ \left( \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} \left\| \sum_k c_{\alpha k} \right\|^2 \right)^{1/2} &\leq \sum_k \left( \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} \|c_{\alpha k}\|^2 \right)^{1/2} \end{aligned}$$

*Proof.* This is a Cauchy-Schwarz inequality and a triangle inequality.

We now define a few abbreviations that depend implicitly on a function  $u \in \mathcal{C}^\infty$  and on a number  $z \in \mathbb{C}$ . As before,  $u_\alpha = \partial^\alpha u$ . For all  $\ell \in \mathbb{N}_0$  set

$$\begin{aligned} \mathcal{X}_\ell &= \left( \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} \|[\partial^\alpha, \mathbf{D}_z]u\|^2 \right)^{1/2} \\ \mathcal{X}'_\ell &= \left( \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} \|[\partial^\alpha, \mathbf{D}_z]_{\text{lot}} u\|^2 \right)^{1/2} \\ \mathcal{X}''_\ell &= \left( \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} \|A^i u_{\alpha+e_i}\|^2 \right)^{1/2} \end{aligned}$$

For all integers  $0 \leq k \leq \ell$  set

$$\begin{aligned} \mathcal{A}_{\ell k} &= \left( \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} \left\| \sum_{\substack{\beta \leq \alpha \\ |\beta|=k}} \binom{\alpha}{\beta} (\partial^\beta A^i) u_{\alpha-\beta+e_i} \right\|^2 \right)^{1/2} \\ \mathcal{B}_{\ell k} &= \left( \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} \left\| \sum_{\substack{\beta \leq \alpha \\ |\beta|=k}} \binom{\alpha}{\beta} (\partial^\beta (B + zA^0)) u_{\alpha-\beta} \right\|^2 \right)^{1/2} \end{aligned}$$

**Lemma 5.10.** *If  $\text{Re } z \geq z_*$ , then for all  $u \in \mathcal{C}^\infty$  and  $\ell \in \mathbb{N}_0$  we have*

$$\|u\|_\ell \leq \frac{\xi}{\xi R_z + \ell} (\|\mathbf{D}_z u\|_\ell + \mathcal{X}'_\ell) + \frac{\ell |\Xi|_0}{\xi R_z + \ell} \mathcal{X}''_{\ell-1}$$

Here  $|\Xi|_0$  is defined just like  $|A|_0$  is defined in Section 3.

*Proof.* Set  $f = \mathbf{D}_z u$  and  $f_\alpha = \partial^\alpha f$ . Trivially,  $\mathbf{D}_z u_\alpha = f_\alpha - [\partial^\alpha, \mathbf{D}_z]u$ . Since  $\operatorname{Re} z \geq z_*$ , Lemma 5.4 implies

$$R_z \langle u_\alpha, u_\alpha \rangle \leq \operatorname{Re} \langle u_\alpha, f_\alpha - [\partial^\alpha, \mathbf{D}_z]u \rangle$$

Equivalently,

$$R_z \langle u_\alpha, u_\alpha \rangle + \operatorname{Re} \langle u_\alpha, ([\partial^\alpha, \mathbf{D}_z] - [\partial^\alpha, \mathbf{D}_z]_{\text{lot}})u \rangle \leq \operatorname{Re} \langle u_\alpha, f_\alpha - [\partial^\alpha, \mathbf{D}_z]_{\text{lot}}u \rangle$$

Multiply both sides by  $\ell!/\alpha!$  and then sum over  $|\alpha| = \ell$ . On the left hand side, use Lemma 5.7. Then use Lemma 5.9 in a number of places to get

$$\left(R_z + \frac{\ell}{\xi}\right) \|u\|_\ell^2 \leq \|u\|_\ell (\|f\|_\ell + \mathcal{X}'_\ell) + \frac{\ell}{\xi} |\Xi|_0 \|u\|_\ell \mathcal{X}''_{\ell-1}$$

where, for the rightmost term, one also needs

$$\left( \sum_{|\beta|=\ell-1} \frac{(\ell-1)!}{\beta!} \|\Xi^i u_{\beta+e_i}\|^2 \right)^{1/2} \leq |\Xi|_0 \|u\|_\ell$$

which follows from the definition of  $|\Xi|_0$  and Lemma 5.8.

**Lemma 5.11.** *We have  $\mathcal{X}''_\ell \leq \|\mathbf{D}_z u\|_\ell + \mathcal{X}_\ell + \mathcal{B}_{\ell 0}$ .*

*Proof.* Use the triangle inequality in Lemma 5.9, together with

$$\begin{aligned} A^i u_{\alpha+e_i} &= (\mathbf{D}_z - B - zA^0)u_\alpha \\ &= \partial^\alpha (\mathbf{D}_z u) - [\partial^\alpha, \mathbf{D}_z]u - (B + zA^0)u_\alpha \end{aligned}$$

**Lemma 5.12.** *We have<sup>20</sup>*

$$\begin{aligned} \mathcal{X}_\ell + \mathcal{B}_{\ell 0} &\leq \sum_{k=1}^{\ell} \mathcal{A}_{\ell k} + \sum_{k=0}^{\ell} \mathcal{B}_{\ell k} \\ \mathcal{X}'_\ell &\leq \sum_{k=2}^{\ell} \mathcal{A}_{\ell k} + \sum_{k=1}^{\ell} \mathcal{B}_{\ell k} \end{aligned}$$

*Proof.* Use the triangle inequality in Lemma 5.9. More in detail, every summation  $\sum_{\beta \leq \alpha}$  as in Lemma 5.6 is written as  $\sum_k \sum_{\beta \leq \alpha, |\beta|=k}$ , and then the triangle inequality is used for the summation over  $k$ .

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<sup>20</sup>The summation  $\sum_{k=K}^{\ell}$  gives zero when  $\ell < K$ .

**Lemma 5.13.** *If  $\operatorname{Re} z \geq z_*$ , then for all  $u \in \mathcal{C}^\infty$  and  $\ell \in \mathbb{N}_0$  we have*

$$\begin{aligned} \|u\|_\ell &\leq \frac{\xi}{\xi R_z + \ell} \left( \|\mathbf{D}_z u\|_\ell + \sum_{k=2}^{\ell} \mathcal{A}_{\ell k} + \sum_{k=1}^{\ell} \mathcal{B}_{\ell k} \right) \\ &\quad + \frac{\ell |\Xi|_0}{\xi R_z + \ell} \left( \|\mathbf{D}_z u\|_{\ell-1} + \sum_{k=1}^{\ell-1} \mathcal{A}_{\ell-1,k} + \sum_{k=0}^{\ell-1} \mathcal{B}_{\ell-1,k} \right) \end{aligned}$$

*Proof.* Lemmas 5.10, 5.11, 5.12. The special case  $\ell = 0$ , which simplifies to  $\|u\| \leq R_z^{-1} \|\mathbf{D}_z u\|$ , is also a direct corollary of Lemma 5.4.

**Lemma 5.14.** *For all integers  $0 \leq k \leq \ell$ ,*

$$\begin{aligned} \mathcal{A}_{\ell k} &\leq \binom{\ell}{k}^{1/2} \binom{\ell+n}{k}^{1/2} |A|_k \|u\|_{\ell-k+1} \\ \mathcal{B}_{\ell k} &\leq \binom{\ell}{k}^{1/2} \binom{\ell+n}{k}^{1/2} |B + zA^0|_k \|u\|_{\ell-k} \end{aligned}$$

*Proof.* The definitions of  $\mathcal{A}_{\ell k}$  and  $|A|_k$  imply

$$\mathcal{A}_{\ell k} \leq |A|_k \left( \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} \sum_{\substack{\beta \leq \alpha \\ |\beta|=k}} \sum_{i=0}^n \frac{\beta!}{k!} \binom{\alpha}{\beta}^2 \|u_{\alpha-\beta+e_i}\|^2 \right)^{1/2}$$

It is convenient to replace  $\|u_{\alpha-\beta+e_i}\|^2 = \sum_{|\gamma|=\ell-k+1} \|u_\gamma\|^2 \delta_{\alpha-\beta+e_i=\gamma}$  and to move the summation over  $\gamma$  to the left, which gives

$$\mathcal{A}_{\ell k} \leq |A|_k \left( \sum_{|\gamma|=\ell-k+1} c_\gamma \frac{(\ell-k+1)!}{\gamma!} \|u_\gamma\|^2 \right)^{1/2}$$

with the purely combinatorial coefficient

$$c_\gamma = \frac{\gamma!}{(\ell-k+1)!} \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} \sum_{\substack{\beta \leq \alpha \\ |\beta|=k}} \sum_{i=0}^n \frac{\beta!}{k!} \binom{\alpha}{\beta}^2 \delta_{\alpha-\beta+e_i=\gamma}$$

One can check that  $c_\gamma = \binom{\ell}{k} \binom{\ell+n}{k}$  if  $|\gamma| = \ell - k + 1$ , which is independent of  $\gamma$ , and this gives the desired estimate for  $\mathcal{A}_{\ell k}$ . Similar for  $\mathcal{B}_{\ell k}$ .



**Lemma 5.15.** For  $K = 0, 1$  we have

$$\begin{aligned} \sum_{\ell=0}^{\infty} \frac{q_{\ell}}{(\ell+1)!} \sum_{k=K+1}^{\ell} \mathcal{A}_{\ell k} &\leq Qq_K \|u\|_0 \\ \sum_{\ell=0}^{\infty} \frac{q_{\ell}}{\ell!} \sum_{k=K}^{\ell} \mathcal{B}_{\ell k} &\leq Qq_K (1 + |z|) \|u\|_0 \end{aligned}$$

*Proof.*

- For the 1st estimate, use  $q_{\ell} \leq q_{k-1} q_{\ell-k+1}$  where  $1 \leq k \leq \ell$ .  
For the 2nd estimate, use  $q_{\ell} \leq q_k q_{\ell-k}$  where  $0 \leq k \leq \ell$ .
- Use Lemma 5.14 with  $\binom{\ell}{k}^{1/2} \binom{\ell+n}{k}^{1/2} \leq \binom{\ell}{k} (k+1)^{n/2}$  when  $0 \leq k \leq \ell$ .
- For the 1st estimate, use  $\frac{1}{(\ell+1)!} \binom{\ell}{k} \leq \frac{1}{k! (\ell-k+1)!}$ .  
For the 2nd estimate, use  $\frac{1}{\ell!} \binom{\ell}{k} \leq \frac{1}{k! (\ell-k)!}$ .
- $\sum_{\ell=0}^{\infty} \sum_{k=K}^{\ell} c_{k, \ell-k} = \sum_{k=K}^{\infty} \sum_{p=0}^{\infty} c_{k, p}$  for all  $c : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow [0, \infty)$ .
- Use assumption (iv).

**Lemma 5.16.** Define  $q_{1*} > 0$  by

$$Qq_{1*} (\xi + 3R^{-1} + |\Xi|_0 + 2R^{-1}\xi^{-1}|\Xi|_0) = \frac{1}{2}$$

If  $\operatorname{Re} z \geq z_*$  and  $q_1 \leq q_{1*}$  then, for all  $u \in \mathcal{C}^{\infty}$ ,

$$\|u\|_0 \leq 2e^{2q_1 |\operatorname{Im} z|} (\xi + R^{-1} + |\Xi|_0 q_1) \|\mathbf{D}_z u\|_1$$

*Proof.* We show that the inequality holds without the exponential factor when  $|\operatorname{Im} z| \leq 1$ ; the general case then follows from  $\mathbf{D}_{z+i} = e^{-ix^0} \mathbf{D}_z e^{ix^0}$  and Lemma 5.17. Lemma 5.13 implies

$$\begin{aligned} \|u\|_{\ell} &\leq (\xi + R^{-1}) \frac{1}{\ell+1} \left( \|\mathbf{D}_z u\|_{\ell} + \sum_{k=2}^{\ell} \mathcal{A}_{\ell k} \right) + \frac{1}{R_z} \sum_{k=1}^{\ell} \mathcal{B}_{\ell k} \\ &\quad + |\Xi|_0 \left( \|\mathbf{D}_z u\|_{\ell-1} + \sum_{k=1}^{\ell-1} \mathcal{A}_{\ell-1, k} \right) + \frac{\ell |\Xi|_0}{\xi R_z} \sum_{k=0}^{\ell-1} \mathcal{B}_{\ell-1, k} \end{aligned}$$

Multiply both sides by  $q_{\ell}/\ell!$  and then sum over  $\ell$ . Lemma 5.15 and the fact that  $(1 + |z|)R_z^{-1} \leq 2R^{-1}$ , because we are assuming  $|\operatorname{Im} z| \leq 1$ , imply

$$\begin{aligned} \|u\|_0 &\leq (\xi + R^{-1}) (\|\mathbf{D}_z u\|_1 + Qq_1 \|u\|_0) + 2R^{-1} Qq_1 \|u\|_0 \\ &\quad + |\Xi|_0 (q_1 \|\mathbf{D}_z u\|_1 + Qq_1 \|u\|_0) + 2R^{-1} \xi^{-1} |\Xi|_0 Qq_1 \|u\|_0 \end{aligned}$$

using  $q_\ell \leq q_1 q_{\ell-1}$  and  $q_0 = 1$ . The assumption  $q_1 \leq q_{1*}$  makes the coefficient of  $\|u\|_0$  on the right hand side  $\leq \frac{1}{2}$ . Solve for  $\|u\|_0$  and be done<sup>21</sup>.

**Lemma 5.17.** *The two operators  $e^{\pm ix^0} : \mathcal{H}_h \rightarrow \mathcal{H}_h$  are bounded for all  $h \in \mathbb{N}_0$ . Explicitly,  $\|e^{\pm ix^0} u\|_h \leq e^{q_1} \|u\|_h$  for all  $u \in \mathcal{C}^\infty$ .*

*Proof.* By routine estimation,  $\|e^{\pm ix^0} u\|_\ell \leq \sum_{m=0}^\ell \binom{\ell}{m} \|u\|_{\ell-m}$ . Then use the definition of  $\|\cdot\|_h$  and  $\frac{1}{(\ell+h)!} \binom{\ell}{m} \leq \frac{1}{m!(\ell-m+h)!}$  and  $q_\ell \leq (q_1)^m q_{\ell-m}$ .

**Lemma 5.18.** *The operator  $A^0 : \mathcal{H}_h \rightarrow \mathcal{H}_h$  is bounded for all  $h \in \mathbb{N}_0$ .*

*Proof.* Note that  $\|(B + zA^0)u\|_\ell \leq \sum_{k=0}^\ell \mathcal{B}_{\ell k}$ . Similar to Lemma 5.15,

$$\sum_{\ell=0}^\infty \frac{q_\ell}{(\ell+h)!} \sum_{k=0}^\ell \mathcal{B}_{\ell k} \leq Q(1+|z|) \|u\|_h$$

using  $\frac{1}{(\ell+h)!} \binom{\ell}{k} \leq \frac{1}{k!(\ell-k+h)!}$ . Hence  $\|(B + zA^0)u\|_h \leq Q(1+|z|) \|u\|_h$ . Hence the map  $B + zA^0 : \mathcal{H}_h \rightarrow \mathcal{H}_h$  is bounded for all  $z \in \mathbb{C}$ , hence  $A^0$  is.

**Lemma 5.19.** *Let  $\widehat{\Omega}_{x^0 \geq 0} \subseteq \widehat{\Omega}$  be the subset of the universal cover space corresponding to  $x^0 \geq 0$ . Suppose  $\operatorname{Re} z \geq z_*$  and suppose*

$$\widehat{u} : \widehat{\Omega}_{x^0 \geq 0} \rightarrow \mathbb{C}^N$$

*is  $\infty$ -differentiable and satisfies  $\widehat{\mathbf{D}}_z \widehat{u} = 0$ . Then each derivative  $\widehat{u}_\alpha = \partial^\alpha \widehat{u}$  converges to zero exponentially fast as  $x^0 \rightarrow +\infty$ , uniformly in  $x^1, \dots, x^n$ .*

*Proof.* Set

$$E_\ell(x^0) = \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} \int_{\text{unit ball in } \mathbb{R}^n} \left( \frac{1}{2} \widehat{u}_\alpha^\dagger A^0 \widehat{u}_\alpha \right)_{\text{at time } x^0}$$

It suffices to show that for all  $\ell \in \mathbb{N}_0$  and all  $0 \leq c < 1$ ,

$$\sup_{x^0 \geq 0} |e^{cx^0} E_\ell(x^0)| < \infty$$

The proof is by induction over  $\ell$ . Since  $\operatorname{Re} z \geq z_*$ , the first inequality in Lemma 5.3<sup>22</sup> and the divergence theorem and (ii) imply

$$\frac{d}{dx^0} E_\ell(x^0) \leq -E_\ell(x^0) + \operatorname{Re} \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} \int_{\text{unit ball in } \mathbb{R}^n} \left( \widehat{u}_\alpha^\dagger \widehat{\mathbf{D}}_z \widehat{u}_\alpha \right)_{\text{at time } x^0}$$

<sup>21</sup>Actually,  $\|u\|_0$  could be infinite, and then the proof does not work. However, if  $q_\ell = 0$  for one  $\ell$  and hence almost all  $\ell$ , then  $\|u\|_0 < \infty$  and the proof does work. The general case then follows from truncating the sequence  $(q_\ell)$  at some index and taking the limit.

<sup>22</sup>This lemma on  $\Omega$  also holds on the universal cover  $\widehat{\Omega}$ .

In the second term on the right hand side, replace<sup>23</sup>

$$\widehat{\mathbf{D}}_z \widehat{u}_\alpha = -([\partial^\alpha, \widehat{\mathbf{D}}_z] - [\partial^\alpha, \widehat{\mathbf{D}}_z]_{\text{lot}}) \widehat{u} - [\partial^\alpha, \widehat{\mathbf{D}}_z]_{\text{lot}} \widehat{u}$$

Now use Lemma 5.7<sup>24</sup>. The first term coming from Lemma 5.7 has favorable sign and is dropped. In the second term coming from Lemma 5.7 we replace  $A^j \widehat{u}_{\beta+e_j} = [\widehat{\mathbf{D}}_z, \partial^\beta] \widehat{u} - (B + zA^0) \widehat{u}_\beta$  where always  $|\beta| = \ell - 1$ . Now triangle and Cauchy-Schwarz inequalities imply

$$\frac{d}{dx^0} E_\ell \leq -E_\ell + (\text{const}) (E_0^{1/2} + \dots + E_{\ell-1}^{1/2}) E_\ell^{1/2}$$

given that  $A^0 > 0$  holds uniformly by (i). The unspecified constant depends on many things, including  $\ell$ , but it does not depend on  $x \in \widehat{\Omega}$ . We get

$$\frac{d}{dx^0} (e^{cx^0/2} E_\ell^{1/2}) \leq (\text{const}) e^{cx^0/2} (E_0^{1/2} + \dots + E_{\ell-1}^{1/2})$$

whenever  $0 \leq c < 1$ . By the induction hypothesis, the integral of the right hand side over  $x^0 \in [0, \infty)$  is finite, and the lemma follows.

**Lemma 5.20.** *If  $\text{Re } z \geq z_*$  then  $\mathbf{D}_z : \mathcal{C}^\infty \rightarrow \mathcal{C}^\infty$  is surjective.*

*Proof.* Given any  $f \in \mathcal{C}^\infty$ , we construct a  $u \in \mathcal{C}^\infty$  such that  $\mathbf{D}_z u = f$ . Let  $\widehat{f} : \widehat{\Omega} \rightarrow \mathbb{C}^N$  be the lift of  $f$  to the universal cover. Let

$$\widehat{u} : \widehat{\Omega}_{x^0 \geq 0} \rightarrow \mathbb{C}^N$$

be the unique  $\infty$ -differentiable solution to  $\widehat{\mathbf{D}}_z \widehat{u} = \widehat{f}$  that vanishes at  $x^0 = 0$ <sup>25</sup>. Since  $\widehat{f} \circ \mathbf{T} = \widehat{f}$  we have  $\widehat{\mathbf{D}}_z(\widehat{u} - \widehat{u} \circ \mathbf{T}) = 0$ . By  $\text{Re } z \geq z_*$  and Lemma 5.19 we have  $\widehat{u} - \widehat{u} \circ \mathbf{T} \rightarrow 0$  exponentially fast as  $x^0 \rightarrow +\infty$ ; same for all derivatives of all orders. Hence  $\widehat{u} \circ \mathbf{T}^p - \widehat{u} \circ \mathbf{T}^{p+1} \rightarrow 0$  exponentially fast as  $p \rightarrow \infty$  on the compact  $\widehat{\Omega}_{0 \leq x^0 \leq 4\pi}$ . This being exponentially fast, we have  $\widehat{u} \circ \mathbf{T}^p - \widehat{u} \circ \mathbf{T}^q \rightarrow 0$  as  $p, q \rightarrow \infty$ . The limit of  $\widehat{u} \circ \mathbf{T}^p$  on  $\widehat{\Omega}_{0 \leq x^0 \leq 4\pi}$  descends to an  $\infty$ -differentiable  $u : \Omega \rightarrow \mathbb{C}^N$ . Since  $\widehat{\mathbf{D}}_z(\widehat{u} \circ \mathbf{T}^p) = \widehat{f}$  for all  $p$ , we have  $\mathbf{D}_z u = f$ .

## 6 Finite codimension stability

This section relies heavily on Theorem 4.3. To be able to freely use this theorem, we always assume in this section:

$$q_1 \leq q_{1*} \text{ and } q_\ell > 0 \text{ for all } \ell$$

<sup>23</sup>Recall from (i) that  $A^i, B$  are  $\infty$ -differentiable, hence the commutator is defined.

<sup>24</sup>This lemma on  $\Omega$  also holds on the universal cover  $\widehat{\Omega}$ . Recall that it relies on (iii).

<sup>25</sup>Existence and uniqueness for linear symmetric hyperbolic systems. Use (i) and (ii).

This assumption is repeated explicitly in Theorem 6.1 below, for emphasis, but the assumption is implicit throughout this section.

Define the following function spaces:

- $\widehat{\mathcal{C}}^\infty = \{\widehat{u} : \widehat{\Omega} \rightarrow \mathbb{C}^N \mid \widehat{u} \text{ is } \infty\text{-differentiable}\}.$
- For all fixed  $a < b$ , denote by  $\widehat{\mathcal{H}}_{1,a,b}$  the set of all  $\widehat{u} \in \widehat{\mathcal{C}}^\infty$  such that

$$\sum_{\ell=0}^{\infty} \frac{q_\ell}{(\ell+1)!} \left( \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} \int_{\widehat{\Omega}} e^{-2cx^0} |\partial^\alpha \widehat{u}|^2 \right)^{1/2} < \infty$$

for each  $c \in \mathbb{R}$  with  $a < c < b$ .

Roughly, a function is in  $\widehat{\mathcal{H}}_{1,-\infty,\infty}$  if it is smooth as given by the sequence  $(q_\ell)$ , and if it and all its derivatives of all orders decay super-exponentially for both  $x^0 \rightarrow \pm\infty$ . This space allows us to state a clean theorem; one can certainly relax the assumptions of the theorem in various directions.

**Theorem 6.1** (Finite codimension stability). *In addition to the abstract assumptions in Section 3, assume  $q_1 \leq q_{1*}$  and  $q_\ell > 0$  for all  $\ell$ . Let*

$$\widehat{\mathbf{D}}_{\text{ret}}^{-1} : \widehat{\mathcal{H}}_{1,-\infty,\infty} \rightarrow \widehat{\mathcal{C}}^\infty$$

*be the retarded Green's function of  $\widehat{\mathbf{D}}$ , explicitly constructed in Lemma 6.5. There exists a finite-rank operator<sup>26</sup>*

$$\mathbf{F} : \widehat{\mathcal{H}}_{1,-\infty,\infty} \rightarrow \widehat{\mathcal{C}}^\infty$$

*such that  $\widehat{\mathbf{D}} \circ \mathbf{F} = 0$  and such that all elements in the image of*

$$\widehat{\mathbf{D}}_{\text{ret}}^{-1} - \mathbf{F}$$

*decay exponentially fast as  $x^0 \rightarrow +\infty$ , uniformly in  $x^1, \dots, x^n$ , and the same decay statement holds for all partial derivatives of all orders.*

In the remainder of this section, we prove this theorem and provide details, including an explicit formula for  $\mathbf{F}$  and a description of its image.

**Lemma 6.2.** *For all  $\widehat{u} \in \widehat{\mathcal{H}}_{1,a,b}$ :*

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<sup>26</sup>If  $V$  and  $V'$  are vector spaces, then a linear operator  $V \rightarrow V'$  is a finite-rank operator iff it is the composition of a linear map  $V \rightarrow \mathbb{C}^J$  with a linear map  $\mathbb{C}^J \rightarrow V'$ , for some  $J < \infty$ . The maps should be continuous, but for simplicity, we do not introduce topologies.

- If  $a < 0$  then  $\widehat{u}$  decays exponentially fast as  $x^0 \rightarrow +\infty$ .
- If  $b > 0$  then  $\widehat{u}$  decays exponentially fast as  $x^0 \rightarrow -\infty$ .
- If  $a = -\infty$  then  $\widehat{u}$  decays super-exponentially fast as  $x^0 \rightarrow +\infty$ .
- If  $b = +\infty$  then  $\widehat{u}$  decays super-exponentially fast as  $x^0 \rightarrow -\infty$ .

The same decay statements hold for all partial derivatives of all orders of  $\widehat{u}$ .

Define  $S_{a,b} = \{z \in \mathbb{C} \mid a < \operatorname{Re} z < b\}$ , an infinite vertical strip.

**Lemma 6.3** (Fourier-like transform). *In the following, let  $u_z$  be the evaluation of the map  $u_*$  at the point  $z \in S_{a,b}$ . There is a linear, bijective map*

$$\left\{ u_* : S_{a,b} \rightarrow \mathcal{H}_1 \left| \begin{array}{l} u_* \text{ is holomorphic} \\ u_{z+i} = e^{-ix^0} u_z \text{ for all } z \in S_{a,b} \end{array} \right. \right\} \rightarrow \widehat{\mathcal{H}}_{1,a,b}$$

given by  $u_* \mapsto \widehat{u}$  where

$$\widehat{u}(x) = \frac{1}{i} \int_{z'}^{z'+i} dz e^{zx^0} u_z(x)$$

for all  $x \in \widehat{\Omega}$ , all  $z' \in S_{a,b}$  and all paths contained in  $S_{a,b}$ . *Convention: From here on, it is implicit that  $u_*$  denotes the transform of  $\widehat{u}$  and conversely.*

*Proof.* The integral does not depend on the choice of path, because the integrand is holomorphic in  $z \in S_{a,b}$  and periodic under  $z \sim z + i$ . For injectivity, use a straight path  $t \mapsto c + it$  with  $a < c < b$  to get

$$\widehat{u}(x^0 + 2\pi p, x^1, \dots, x^n) = \int_0^1 dt e^{(c+it)x^0} e^{(c+it)2\pi p} u_{c+it}(x)$$

for all  $p \in \mathbb{Z}$ , and now note that if  $\widehat{u} = 0$  then  $\mathbb{R}/\mathbb{Z} \rightarrow \mathbb{C}$ ,  $t \mapsto e^{(c+it)x^0} u_{c+it}(x)$  is a function all whose Fourier coefficients vanish, hence  $u_* = 0$ . For surjectivity, one checks that every  $\widehat{u} \in \widehat{\mathcal{H}}_{1,a,b}$  is the image point of  $u_*$  given by

$$u_z = \sum_{p \in \mathbb{Z}} (e^{-zx^0} \widehat{u}) \circ \mathbf{T}^p$$

Apart from estimates that we omit<sup>27</sup>, this concludes the proof.

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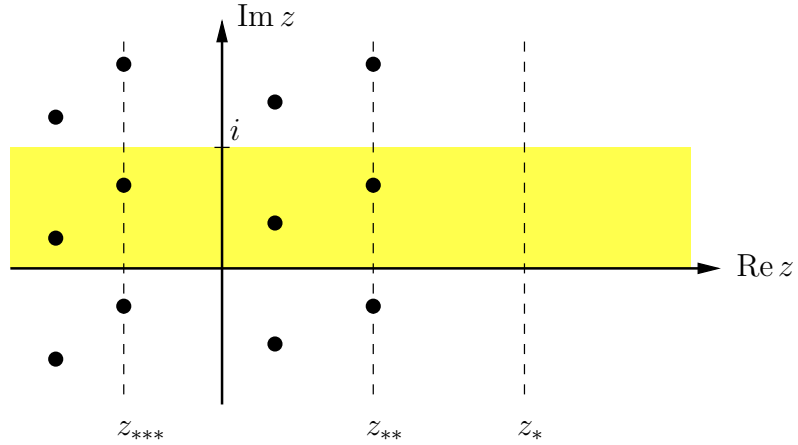
<sup>27</sup>Useful lemma: For all  $z \in \mathbb{C}$  and  $\widehat{u} \in \widehat{\mathcal{C}}^\infty$  and all ‘weight’ functions  $w : \widehat{\Omega} \rightarrow [0, \infty)$ :

$$\begin{aligned} \sum_{\ell=0}^{\infty} \frac{q_\ell}{(\ell+1)!} \left( \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} \int_{\widehat{\Omega}} w |\partial^\alpha (e^{zx^0} \widehat{u})|^2 \right)^{1/2} \\ \leq e^{q_1|z|} \sum_{\ell=0}^{\infty} \frac{q_\ell}{(\ell+1)!} \left( \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} \int_{\widehat{\Omega}} w e^{2(\operatorname{Re} z)x^0} |\partial^\alpha \widehat{u}|^2 \right)^{1/2} \end{aligned}$$

We now define  $z_{***} \leq z_{**} \leq z_*$  with  $z_{***} < 0$  by

$$\begin{aligned} z_{**} &= \sup \{ \operatorname{Re} z \mid z \text{ is a pole} \} \\ z_{***} &= \sup \{ \operatorname{Re} z \mid z \text{ is a pole with } \operatorname{Re} z < 0 \} \end{aligned}$$

where the poles are those of the map  $z \mapsto \mathbf{D}_z^{-1}$  in Theorem 4.3<sup>28</sup>. The set of poles is periodic under  $z \sim z + i$ . Here is an example:



The domain indicated in this figure,  $\{z \in \mathbb{C} \mid 0 \leq \operatorname{Im} z < 1\}$ , is a fundamental domain for  $z \sim z + i$ . Here exactly two poles have nonnegative real part in a fundamental domain, but in general there could be any finite number.

**Lemma 6.4.** *For all  $z \in \mathbb{C}$  away from the poles of  $z \mapsto \mathbf{D}_z^{-1}$ , and for all  $f \in \mathcal{H}_1$ , applying  $\widehat{\mathbf{D}}$  to the function  $x \mapsto e^{zx^0}(\mathbf{D}_z^{-1}f)(x)$  gives  $x \mapsto e^{zx^0}f(x)$ .*

*Proof.* If  $\operatorname{Re} z \geq z_*$  then  $\mathbf{D}_z^{-1}$  is an honest inverse in the sense of Theorem 4.1 and the claim follows from the operator identity  $\widehat{\mathbf{D}}e^{zx^0} = e^{zx^0}(\widehat{\mathbf{D}} + zA^0)$ . This implies the claim for general  $z$  by a meromorphic continuation argument.

**Lemma 6.5.** *Define a linear map*

$$\widehat{\mathbf{D}}_{\text{ret}}^{-1} : \widehat{\mathcal{H}}_{1,-\infty,\infty} \rightarrow \widehat{\mathcal{H}}_{1,z_{**},\infty}$$

*by  $\widehat{f} \mapsto \widehat{u}$  where  $u_z = \mathbf{D}_z^{-1}f_z$ , using Lemma 6.3. Explicitly,*

$$(\widehat{\mathbf{D}}_{\text{ret}}^{-1}\widehat{f})(x) = \frac{1}{i} \int_{z'}^{z'+i} dz e^{zx^0} (\mathbf{D}_z^{-1}f_z)(x)$$

*for all  $x \in \widehat{\Omega}$  and all paths contained in  $S_{z_{**},\infty}$ . Then this is a right-inverse of  $\widehat{\mathbf{D}}$ , and more specifically, it is the retarded Green's function.*

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<sup>28</sup>We write sup (rather than max) only because there could be no poles at all.

*Proof.* It is a right-inverse by Lemma 6.4. The map  $\widehat{\mathbf{D}}_{\text{ret}}^{-1}$  is the retarded Green's function because all elements in its image converge to zero super-exponentially as  $x^0 \rightarrow -\infty$ , and so do all partial derivatives of all orders.

We now define

$$\Lambda = \{z \mid z \text{ is a pole with } \operatorname{Re} z \geq 0\} \cap \{z \mid 0 \leq \operatorname{Im} z < 1\}$$

The second factor is a fundamental domain for  $z \sim z + i$ , and any other fundamental domain could be used instead. This is always a finite set,  $|\Lambda| < \infty$ .

**Lemma 6.6.** Define  $\mathbf{F} : \widehat{\mathcal{H}}_{1,-\infty,\infty} \rightarrow \widehat{\mathcal{C}}^\infty$  by

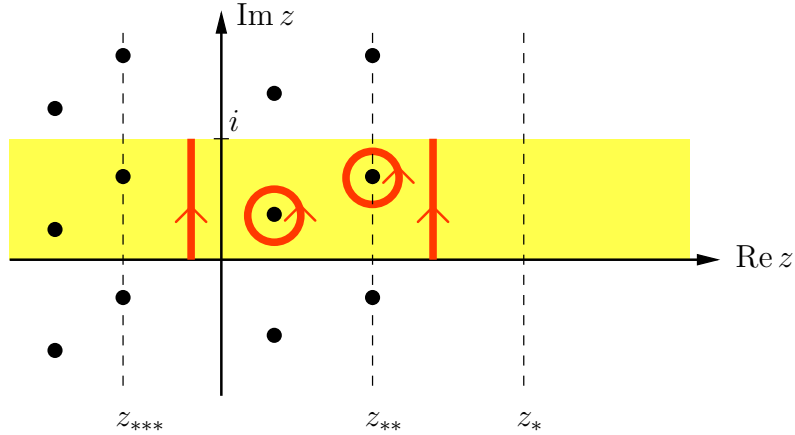
$$(\mathbf{F}\widehat{f})(x) = \sum_{\lambda \in \Lambda} \frac{1}{i} \int_{\text{loop about } \lambda} dz e^{zx^0} (\mathbf{D}_z^{-1} f_z)(x)$$

for all  $x \in \widehat{\Omega}$ . Then

$$\begin{aligned} \widehat{\mathbf{D}} \circ \mathbf{F} &= 0 \\ \text{image}(\widehat{\mathbf{D}}_{\text{ret}}^{-1} - \mathbf{F}) &\subseteq \widehat{\mathcal{H}}_{1,z_{***},0} \end{aligned}$$

*Note:* The decay in Theorem 6.1 now follows from  $z_{***} < 0$  and Lemma 6.2.

*Proof.* For  $\widehat{\mathbf{D}} \circ \mathbf{F} = 0$  use Lemma 6.4 and  $\int_{\text{loop about } \lambda} dz e^{zx^0} f_z(x) = 0$ . For the image, the case  $z_{**} < 0$  is trivial, because on the one hand  $z_{**} = z_{***}$  and hence  $\text{image}(\widehat{\mathbf{D}}_{\text{ret}}^{-1}) \subseteq \widehat{\mathcal{H}}_{1,z_{***},0}$ , and on the other hand  $|\Lambda| = 0$  and  $\mathbf{F} = 0$ . The case  $z_{**} \geq 0$  is conveniently discussed using our  $|\Lambda| = 2$  example:



Let  $\widehat{f} \in \widehat{\mathcal{H}}_{1,-\infty,\infty}$ . We want to show that  $(\widehat{\mathbf{D}}_{\text{ret}}^{-1} - \mathbf{F})\widehat{f} \in \widehat{\mathcal{H}}_{1,z_{***},0}$ . Fix  $x \in \widehat{\Omega}$ . Then  $I_z(x) = i^{-1} e^{zx^0} (\mathbf{D}_z^{-1} f_z)(x)$  is meromorphic in  $z$  with poles coming from  $\mathbf{D}_z^{-1}$  only, and periodic under  $z \sim z + i$ . Observe that:

- The integral of  $I_z(x)$  along the path on the right gives  $(\widehat{\mathbf{D}}_{\text{ret}}^{-1}\widehat{f})(x)$ .
- The integral of  $I_z(x)$  about the  $|\Lambda|$ -many loops gives  $(\mathbf{F}\widehat{f})(x)$ .

Their difference is equal to the integral of  $I_z(x)$  along the path on the left, by Cauchy's theorem, which yields an element of  $\widehat{\mathcal{H}}_{1,z^{***},0}$  by Lemma 6.3.

**Lemma 6.7.** *If  $\lambda \in \Lambda$  and  $\ell \in \mathbb{N}_0$  then*

$$(P_{\lambda\ell}f)(x) = \frac{1}{2\pi i} \int_{\text{loop about } \lambda} dz (z - \lambda)^\ell (\mathbf{D}_z^{-1}f)(x)$$

*is a finite-rank operator  $P_{\lambda\ell} : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ , and*

$$(\widehat{P}_{\lambda\ell}f)(x) = \frac{1}{2\pi i} \int_{\text{loop about } \lambda} dz (z - \lambda)^\ell e^{zx^0} (\mathbf{D}_z^{-1}f)(x)$$

*is a finite-rank operator  $\widehat{P}_{\lambda\ell} : \mathcal{H}_1 \rightarrow \widehat{\mathcal{C}}^\infty$ . We have*

$$\begin{aligned} \widehat{\mathbf{D}} \circ \widehat{P}_{\lambda\ell} &= 0 \\ \text{image}(\widehat{P}_{\lambda\ell}) &\subseteq (\text{polynomials in } x^0) e^{\lambda x^0} \mathcal{H}_1 \end{aligned}$$

*Proof.* We use Theorem 4.3. Since  $\mathbf{D}_z^{-1}$  is compact away from poles,  $P_{\lambda\ell}$  is compact<sup>29</sup>. By the first resolvent identity,  $P_{\lambda k} A^0 P_{\lambda\ell} = P_{\lambda(k+\ell)}$ . It follows that  $\text{image}(P_{\lambda\ell}) \subseteq \text{image}(P_{\lambda 0} A^0)$ . It also follows that  $P_{\lambda 0} A^0$  is a projection. As a compact projection,  $\dim(\text{image}(P_{\lambda 0} A^0)) < \infty$ . Hence  $\dim(\text{image}(P_{\lambda\ell})) < \infty$ , as claimed. Now use  $e^{zx^0} = \sum_{k=0}^{\infty} \frac{1}{k!} (z - \lambda)^k (x^0)^k e^{\lambda x^0}$  which implies

$$\widehat{P}_{\lambda\ell} = \sum_{k=0}^{\infty} \frac{1}{k!} (x^0)^k e^{\lambda x^0} P_{\lambda(k+\ell)}$$

The sum is finite, because  $P_{\lambda k} = 0$  if  $k$  is equal to or bigger than the order of the pole at  $\lambda$ , and therefore  $\widehat{P}_{\lambda\ell}$  is a finite-rank operator as well.

**Lemma 6.8.** *For all  $\widehat{f} \in \widehat{\mathcal{H}}_{1,-\infty,\infty}$  we have*

$$\mathbf{F}\widehat{f} = 2\pi \sum_{\lambda \in \Lambda} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \widehat{P}_{\lambda\ell} \left( \frac{d^\ell f_z}{dz^\ell} \right)_{\text{at } z = \lambda}$$

*There are only finitely many pairs  $(\lambda, \ell) \in \Lambda \times \mathbb{N}_0$  for which  $\widehat{P}_{\lambda\ell} \neq 0$ .*

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<sup>29</sup>The compact operators are a closed subspace of the space of bounded operators.



*Proof.* This follows from Lemmas 6.6 and 6.7, using a Taylor expansion of the holomorphic function  $f_*$  about each  $\lambda \in \Lambda$ .

**Corollary 6.9.** *The operator  $\mathbf{F} : \widehat{\mathcal{H}}_{1,-\infty,\infty} \rightarrow \widehat{\mathcal{C}}^\infty$  is finite-rank and*

$$\begin{aligned} \widehat{\mathbf{D}} \circ \mathbf{F} &= 0 \\ \text{image}(\mathbf{F}) &\subseteq \sum_{\lambda \in \Lambda} (\text{polynomials in } x^0) e^{\lambda x^0} \mathcal{H}_1 \end{aligned}$$

## 7 Remarks

*Remark 7.1.* The assumption that we are on the closed unit ball in the  $n$  spatial coordinates is not essential. One can use

$$\Omega = \{(x^0, x^1, \dots, x^n) \in (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^n \mid (x^1, \dots, x^n) \in \Gamma(x^0)\}$$

where  $\Gamma(x^0) \subseteq \mathbb{R}^n$  satisfies  $\Gamma(x^0 + 2\pi) = \Gamma(x^0)$ , all sufficiently smooth.

*Remark 7.2.* This concerns another generalization that would require a lot of careful checking. Introduce an auxiliary positive definite (Riemannian) metric  $h = h_{ij} dx^i \otimes dx^j$  on  $\Omega$ . This paper may correspond to the special case  $h_{ij} = \delta_{ij}$  of a more general, more geometric formulation of the assumptions and theorems that is invariant under diffeomorphisms of  $\Omega$ , with the understanding that  $h$  is also transformed. More general, because one now has the additional freedom of choosing  $h$ . One can also try to introduce an auxiliary inner product on  $\mathbb{C}^N$  etc. None of this has been tried.

*Remark 7.3.* In applications to general relativity, one has to deal with gauge freedom. One can gauge-fix and make the equations symmetric hyperbolic, but it would also be interesting to try to reformulate the assumptions and theorems in this paper in a more gauge-invariant way.

*Remark 7.4.* Despite Counterexample 2.3, one can generalize Theorem 4.3 to finite differentiability. One may not get a compact resolvent, and it may not be meromorphic on  $w \in \mathbb{C}$ , but it will be on half-planes  $\text{Re } w > w_*$ ; the question is how negative one can take  $w_*$  as a function of  $L = \sup\{\ell \mid q_\ell > 0\}$ . The following sketch relies heavily on the operator seminorm  $\|\cdot\|_{\text{nc}}$  discussed separately in Appendix A. Lemmas 5.1 and 5.2 generalize to

$$\|\mathcal{H}_0 \hookrightarrow \mathcal{H}_1\|_{\text{nc}} = \frac{1}{L+1}$$

which is proved using Rellich's theorem. Informally: the bigger  $L$ , the closer the inclusion operator is to being compact. Using Lemma 5.16, the first part

of Theorem 4.3 generalizes to: If  $\operatorname{Re} z \geq z_*$  and  $q_1 \leq q_{1*}$  then

$$\|\mathbf{D}_z^{-1} : \mathcal{H}_1 \rightarrow \mathcal{H}_1\|_{\text{nc}} \leq \frac{2e^{2q_1|\operatorname{Im} z|}(\xi + R^{-1} + |\Xi|_0 q_1)}{L + 1}$$

The numerator is written out for clarity; here we consider the situation where all parameters in the numerator are fixed. For  $L = \infty$  we recover the original Theorem 4.3, but we are now interested in  $L < \infty$ . Putting things together, including Appendix A, one finds that the bigger  $L$ , the more negative one can take  $w_*$ . As  $L \rightarrow \infty$  one can take  $w_* \rightarrow -\infty$ .

## Acknowledgments

Thanks to Eugene Trubowitz for his interest and many useful suggestions.

## A Non-compactness seminorm

For a bounded operator  $T$  on a Banach space, the operator norm tells one that the resolvent  $(T - \lambda)^{-1}$  exists on  $|\lambda| > \|T\|$ . By contrast, the operator seminorm defined below tells one that the resolvent exists on  $|\lambda| > \|T\|_{\text{nc}}$ , except for a discrete set of points that are actual eigenvalues.

The spectral theory of compact operators is obtained as a special case, because  $\|T\|_{\text{nc}} = 0$  iff  $T$  is compact. In fact, the arguments in this appendix are minor adaptations of standard arguments in Riesz's spectral theory of compact operators. Everything in this appendix is probably available in the literature on 'measures of non-compactness'.

For all Banach spaces  $V, V'$  and all bounded linear  $T : V \rightarrow V'$ , set

$$\|T\|_{\text{nc}} = \inf \left\{ \varepsilon > 0 \left| \begin{array}{l} \text{For every sequence } (v_p)_{p \geq 0} \text{ in } V \text{ with diameter } \leq 1, \\ \text{there is a subsequence of } (Tv_p)_{p \geq 0} \text{ in } V' \text{ with diameter } \leq \varepsilon. \end{array} \right. \right\}$$

Here the diameter is the supremum of all pairwise distances. Then:

- $0 \leq \|T\|_{\text{nc}} \leq \|T\|$ .
- $\|\cdot\|_{\text{nc}}$  is a continuous seminorm on the space of bounded operators.
- $\|T\|_{\text{nc}} = 0$  if and only if  $T$  is compact.
- $\|T_1 T_2\|_{\text{nc}} \leq \|T_1\|_{\text{nc}} \|T_2\|_{\text{nc}}$ .
- If  $\dim V = \infty$  then  $\|\mathbb{1}_V\|_{\text{nc}} = 1$ .

For the remainder of this appendix, fix a Banach space  $V$  and a bounded linear operator  $T : V \rightarrow V$ . Define  $\sigma_{\text{eigenvalue}} \subseteq \sigma \subseteq \mathbb{C}$  by:

$$\begin{aligned}\sigma_{\text{eigenvalue}} &= \{\lambda \in \mathbb{C} \mid T - \lambda \text{ is not injective}\} \\ \sigma &= \{\lambda \in \mathbb{C} \mid T - \lambda \text{ is not bijective}\}\end{aligned}$$

For all  $\lambda \notin \sigma$ , the resolvent  $(T - \lambda)^{-1}$  is a bounded operator, by the open mapping theorem. The spectrum  $\sigma$  is a compact subset of  $\mathbb{C}$ .

**Lemma A.1.** *If  $|\lambda| > \|T\|_{\text{nc}}$  then  $\text{image}(T - \lambda)$  is a closed subspace.*

*Proof.* First observe that  $\text{image}(T - \lambda) = \text{image}(T - \lambda)|_C$  where

$$C = \{v \in V \mid \text{dist}(v, \ker(T - \lambda)) \geq \tfrac{1}{2}\|v\|\}$$

Hence it suffices to show that, if  $(v_p)$  is a sequence in  $C$  such that  $((T - \lambda)v_p)$  converges, then  $(v_p)$  has a Cauchy subsequence. We combine<sup>30</sup>:

- The definition of  $\|T\|_{\text{nc}}$ .
- $\|Tv_q - Tv_p\| - |\lambda|\|v_q - v_p\| \rightarrow 0$  as  $q, p \rightarrow \infty$ .

They imply that *there exists a  $0 < \kappa < 1$  such that every subsequence of  $(v_p)$  with diameter  $d$  contains a subsequence with diameter  $\leq \kappa d$* . In fact, every  $\kappa$  with  $\|T\|_{\text{nc}}/|\lambda| < \kappa < 1$  will do. This implies that  $(v_p)$  has a Cauchy subsequence, and that we are done, if  $(v_p)$  has even just one subsequence that has finite diameter ( $\Leftrightarrow$  that is bounded). The remaining case is  $\|v_p\| \rightarrow \infty$ . Then  $((T - \lambda)v_p/\|v_p\|)$  converges to zero. Hence  $(v_p/\|v_p\|)$  has a Cauchy subsequence, by the argument just given, with limit in  $\ker(T - \lambda)$ . But this contradicts  $v_p \in C$ , namely  $\text{dist}(v_p/\|v_p\|, \ker(T - \lambda)) \geq \frac{1}{2}$ , hence  $\|v_p\| \not\rightarrow \infty$ .

**Lemma A.2.** *If  $r > \|T\|_{\text{nc}}$  then there does not exist a sequence  $(\lambda_p)_{p \geq 0}$  of complex numbers, and a sequence  $(V_p)_{p \geq 0}$  of subspaces of  $V$ , such that:*

- $|\lambda_p| \geq r$ .
- $V_p$  is a closed subspace.
- $V_{p \pm 1} \subseteq V_p$  with proper inclusion, and  $(T - \lambda_p)V_p \subseteq V_{p \pm 1}$ .

*The sign  $\pm$  is arbitrary, but it is understood to be the same in both places.*

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<sup>30</sup>The second by  $\|(Tv_q - Tv_p) - \lambda(v_q - v_p)\| \rightarrow 0$  and the reverse triangle inequality.

*Proof.* We prove the minus version; the plus version is similar. Suppose, by contradiction, that such sequences do exist. By Riesz's lemma, there exist unit vectors  $v_p \in V_p$  with  $\text{dist}(V_{p-1}, v_p) \geq \frac{1}{2}$ . Also note that  $TV_p \subseteq V_p$ . For all integers  $0 \leq q < p$  and  $m \geq 1$  we have

$$\|T^m v_q - T^m v_p\| = \|\underbrace{T^m v_q}_{\in V_{p-1}} - \underbrace{(T^m - \lambda_p^m)v_p}_{\in V_{p-1}} - \lambda_p^m v_p\| \geq \frac{1}{2}|\lambda_p|^m \geq \frac{1}{2}r^m$$

The sequence  $(v_p)$  has diameter  $\leq 2$ , whereas every subsequence of  $(T^m v_p)$  has diameter  $\geq \frac{1}{2}r^m$ . Therefore  $\|T^m\|_{\text{nc}} \geq \frac{1}{4}r^m$ . Therefore  $\|T\|_{\text{nc}} \geq (\frac{1}{4})^{1/m}r$ . Since this holds for all  $m$ , we get  $\|T\|_{\text{nc}} \geq r$ , a contradiction.

**Lemma A.3.** *If  $|\lambda| > \|T\|_{\text{nc}}$  and  $\lambda \in \sigma$  then  $\lambda \in \sigma_{\text{eigenvalue}}$ .*

*Proof.* The subspace  $V_p = \text{image}(T - \lambda)^p$  is closed, by induction, using  $V_{p+1} = (T - \lambda)V_p$  and  $\|T|_{V_p}\|_{\text{nc}} \leq \|T\|_{\text{nc}} < |\lambda|$  and Lemma A.1. Suppose, by contradiction, that  $\lambda \in \sigma \setminus \sigma_{\text{eigenvalue}}$ . Then  $V_{p+1} \subseteq V_p$  is proper. This contradicts the plus version of Lemma A.2 with  $r = |\lambda|$  and  $\lambda_p = \lambda$ .

**Lemma A.4.** *If  $r > \|T\|_{\text{nc}}$  then  $\{\lambda \in \sigma_{\text{eigenvalue}} \mid |\lambda| \geq r\}$  is a finite set.*

*Proof.* Suppose, by contradiction, that there exists a sequence  $(\lambda_p)_{p \geq 0}$  of pairwise distinct  $\lambda_p \in \sigma_{\text{eigenvalue}}$  with  $|\lambda_p| \geq r$ . Pick eigenvectors  $v_p \neq 0$  with  $Tv_p = \lambda_p v_p$  and set  $V_p = \text{span}\{v_0, \dots, v_p\}$ . Then  $\dim V_p = p + 1$ , since the  $\lambda_p$  are distinct. This contradicts the minus version of Lemma A.2.

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